# Convex Hulls, Triangulations, and Voronoi Diagrams of Planar Point Sets on the Congested Clique* 

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#### Abstract

We consider geometric problems on planar $n^{2}$-point sets in the congested clique model. Initially, each node in the $n$-clique network holds a batch of $n$ distinct points in the Euclidean plane given by $O(\log n)$-bit coordinates. In each round, each node can send a distinct $O(\log n)$-bit message to each other node in the clique and perform unlimited local computations. We show that the convex hull of the input $n^{2}$-point set can be constructed in $O(\min \{h, \log n\})$ rounds, where $h$ is the size of the hull, on the congested clique. We also show that a triangulation of the input $n^{2}$-point set can be constructed in $O\left(\log ^{2} n\right)$ rounds on the congested clique. Finally, we demonstrate that the Voronoi diagram of $n^{2}$ points with $O(\log n)$-bit coordinates drawn uniformly at random from a unit square can be computed within the square with high probability in $O(1)$ rounds on the congested clique.


Keywords: convex hull, triangulation, Voronoi diagram, distributed algorithms, the congested clique model

## 1 Introduction

The congested clique is a model of communication/computation that focuses on the cost of communication between the nodes in a network and ignores that of local computation within each node. This model was introduced by Lotker et al. [10]. It can be seen as a reaction to the criticized Parallel Random Access Machine (PRAM) model, studied extensively in the 1980s and 1990s, which in contrast focuses on the computation cost and ignores the communication cost 1 .

[^0]In recent decades, the complexity of dense graph problems has been intensively studied in the congested clique model. Typically, each node of the clique network initially represents a distinct vertex of the input graph and knows that vertex's neighborhood in the input graph. The nodes are assumed to have unique numbers (IDs) between 1 and $n$ which are already known by all nodes in the network at the start of the computation. Then, in each round, each of the $n$ nodes can send a distinct message of $O(\log n)$ bits to each other node and can perform unlimited local computation; see Fig. 1. Several dense graph problems, for example, the minimum spanning tree problem, have been shown to admit $O(1)$-round algorithms in the congested clique model [11|15]. Note that when the input graph is of bounded degree, each node can send its whole information to a distinguished node in $O(1)$ rounds. The distinguished node can then solve the graph problem locally. However, when the input graph is dense such a trivial solution requires $\Omega(n)$ rounds.


Fig. 1. An example of a congested clique network.

Researchers have also studied problems not falling in the category of graph problems, like matrix multiplication [3] or sorting and routing [8], in the congested clique model. In both cases, one assumes that the basic items, i.e., matrix entries or keys, respectively, have $O(\log n)$ bit representations and that each node initially has a batch of $n$ such items. As in the graph case, each node can send a distinct $O(\log n)$-bit message to each other node and perform unlimited computation in every round. Significantly, it has been shown that matrix multiplication admits an $O\left(n^{1-2 / \omega}\right)$-round algorithm [3], where $\omega$ is the exponent of fast matrix multiplication, while sorting and routing admit $O(1)$-round algorithms (Theorems 4.5 and 3.7 in [8]) under the aforementioned assumptions.

We extend this approach to include basic geometric problems on planar point sets. These problems are generally known to admit polylogarithmic time solutions on PRAMs with a polynomial number of processors [1]. Initially, each node of the $n$-clique network holds a batch of $n$ points belonging to the input set $S$ of $n^{2}$ points with $O(\log n)$-bit coordinates in the Euclidean plane. As in the graph, matrix, sorting, and routing cases, in each round, each node can send a distinct $O(\log n)$-bit message to each other node and perform unlimited local computa-
tions. Analogously, trivial solutions consisting in gathering the whole data in a distinguished node require $\Omega(n)$ rounds.

More precisely, the problems that we consider are computing the convex hull, a triangulation, and the Voronoi diagram of a set $S$ of $n^{2}$ points with $O(\log n)$-bit coordinates in the plane, defined next. The convex hull of $S$ is the smallest convex polygon $P$ for which every $q \in S$ lies in the interior of $P$ or on the boundary of $P$. A triangulation of $S$ is a maximal set of non-crossing edges between pairs of points from $S$. Finally, the Voronoi diagram of $S$ is the partition of the plane into $|S|$ regions such that each region consists of all points in the plane having the same closest point in $S$.

Our contributions are as follows. First, we provide a simple implementation of the Quick Convex Hull algorithm [5], showing that the convex hull of $S$ can be constructed in $O(h)$ rounds on the congested clique, where $h$ is the size of the hull. Then, we present and analyze a more refined algorithm for the convex hull of $S$ on the congested clique running in $O(\log n)$ rounds. Next, we present a divide-and-conquer method for constructing a triangulation of $S$ in $O\left(\log ^{2} n\right)$ rounds on the congested clique. We conclude with with remarks on the construction of the Voronoi diagram of a planar point set. In particular, we show that the Voronoi diagram of $n^{2}$ points with $O(\log n)$-bit coordinates drawn uniformly at random from a unit square can be computed within the square with high probability in $O(1)$ rounds on the congested clique.

We also refer to the points of the input point set as vertices, while reserving the word nodes to refer to the communicating parties in the underlying congested clique network. In order to simplify the presentation, we assume throughout the paper that the points in the input point sets are in general position.

## 2 Preliminaries

Let $S=\left\{p_{1}, \ldots, p_{n}\right\}$ be a set of $n$ distinct points in the Euclidean plane such that the $x$-coordinate of each point is not smaller than that of $p_{1}$ and not greater than that of $p_{n}$. The upper hull of $S$ (with respect to $\left(p_{1}, p_{n}\right)$ ) is the part of the convex hull of $S$ beginning in $p_{1}$ and ending in $p_{n}$ in clockwise order. Symmetrically, the lower hull of $S$ (with respect to $\left(p_{1}, p_{n}\right)$ ) is the part of the convex hull of $S$ beginning in $p_{n}$ and ending in $p_{1}$ in clockwise order.

A supporting line for the convex hull or upper hull or lower hull of a finite point set in the Euclidean plane is a straight line that touches the hull without crossing it properly. Let $S_{1}, S_{2}$ be two finite sets of points in the Euclidean plane separated by a vertical line. The bridge between the upper (or lower) hull of $S_{1}$ and the upper (or, lower, respectively) hull of $S_{2}$ is a straight line that is a supporting line for both of the upper (lower, respectively) hulls. See Fig. 2 for an illustration.

We define the Information Distribution Task (IDT) [8 as follows:
Each node of the congested $n$-clique holds a set of exactly $n O(\log n)$-bit messages with their destinations, with multiple messages from the same source node to the same destination node allowed. Initially, the destination of each mes-


Fig. 2. An example of the bridge between the upper hulls of $S_{1}$ and $S_{2}$.
sage is known only to its source node. Each node is the destination of exactly $n$ of the aforementioned messages. The messages are globally lexicographically ordered by their source node, their destination, and their number within the source node. For simplicity, each such message explicitly contains these values, in particular making them distinguishable. The goal is to deliver all messages to their destinations, minimizing the total number of rounds.

Lenzen showed that IDT can be solved in 16 rounds (Theorem 3.7 in [8]). He also observed that the relaxed IDT, where each node is required to send and receive at most $n$ messages, easily reduces to IDT in $O(1)$ rounds. Hence, we have the following fact.

Fact 1 [8] The relaxed Information Distribution Task can be solved deterministically within $O(1)$ rounds.

The Sorting Problem (SP) is defined as follows:
Each node $i$ of the congested $n$-clique holds a set of $n O(\log n)$-bit keys. All the keys are different w.l.o.g. Each node $i$ needs to learn all the keys of indices in $[n(i-1)+1, n i]$ (if any) in the total order of all keys.

Lenzen showed that SP can be solved in 37 rounds if each node holds a set of exactly $n$ keys (Theorem 4.5 in [8]). In order to relax the requirement that each node holds exactly $n$ keys to that of with most $n$ keys, we can determine the maximum key and add appropriate different dummy keys in $O(1)$ rounds. Hence, we obtain the following fact.

Fact 2 [8] The relaxed Sorting Problem can be solved in $O(1)$ rounds.

## 3 Quick Convex Hull Algorithm on Congested Clique

The Quick Convex Hull Algorithm (also known as QuickHull or CONVEX) is well known in the literature; see, e.g., [514. Roughly, we shall implement it as follows in the congested clique model. First, the set $S$ of $n^{2}$ input points with $O(\log n)$-bit coordinates is sorted by their $x$-coordinates [8]. As a result, each consecutive clique node gets a consecutive $n$-point fragment of the sorted $S$. Next, each node informs all other nodes about its two extreme points along the
$x$ axis. By using this information, each node can determine the same pair of extreme points $p_{\text {min }}, p_{\max }$ in $S$ along the $x$ axis. Using this extreme pair, each node can decompose its subsequence of $S$ into the upper-hull subsequence consisting of the points that lie above or on the segment $\left(p_{\min }, p_{\max }\right)$ and the lower-hull subsequence consisting of points that lie below or on ( $p_{\min }, p_{\max }$ ). From now on, the upper hull of $S$ and the lower hull of $S$ are computed separately by calling the procedures QuickUpperHull $\left(p_{\min }, p_{\max }\right)$ and QuickLowerHull $\left(p_{\min }, p_{\max }\right)$, respectively. The former procedure proceeds as follows. Each node selects a point $q$ at maximum distance from the segment $\left(p_{\min }, p_{\max }\right)$ among all points in its upper-hull subsequence, excluding the points $p_{\min }$ and $p_{\max }$. Next, it sends the point $q$ to all other nodes. Then, each node selects the same point $q$, different from $p_{\min }$ and $p_{\max }$, at maximum distance from the segment $\left(p_{\min }, p_{\max }\right)$ among all points in the whole upper-hull subsequence; see Fig. 3. Note that $q$ must be a vertex of the upper hull of $S$. Two recursive calls QuickUpper $H u l l\left(p_{\min }, q\right)$ and QuickUpperHull $\left(q, p_{\max }\right)$ follow. The procedure QuickLowerHull is defined symmetrically. As each non-leaf call of these two procedures results in a new vertex of the convex hull, and each step of these procedures but for the recursive calls takes $O(1)$ rounds, the total number of rounds necessary to implement the outlined variant of Quick Convex Hull algorithm, specified in the procedure QuickConvexHull( $S$ ), is proportional to the size of the convex hull of $S$.


Fig. 3. Illustrating the points $p, q, r$ in the procedure QuickUpperHull.
procedure QuickConvexHull(S)
Input: A set of $n^{2}$ points in the Euclidean plane with $O(\log n)$ bit coordinates, each node holds a batch of $n$ input points.
Output: The vertices of the convex hull of $S$ held in clockwise order in consecutive nodes in batches of at most $n$ vertices.

1. Sort the points in $S$ by their $x$-coordinates so each node receives a subsequence consisting of $n$ consecutive points in $S$, in the sorted order.
2. Each node sends the first point and the last point in its subsequence to the other nodes.
3. Each node computes the same point $p_{\max }$ of the maximum $x$-coordinate and the same point $p_{\text {min }}$ of the minimum $x$-coordinate in the whole input sequence $S$ based on the gathered information. (If there are ties in the minimum $x$-coordinate then $p_{\text {min }}$ is set to a point with minimum $y$-coordinate. Similarly, if there are ties in the maximum $x$-coordinate then $p_{\max }$ is set to a point with maximum $y$-coordinate.)
4. Each node decomposes its sorted subsequence into the upper hull subsequence consisting of points above or on the segment connecting $p_{\max }$ and $p_{\text {min }}$ and the lower hull subsequence consisting of the points lying below or on this segment. In particular, the points $p_{\min }$ and $p_{\max }$ are assigned to both upper and lower hull subsequences of the subsequences they belong to.
5. Each node sends its first and last point in its upper hull subsequence as well as its first and last point in its lower hull subsequence to all other nodes.
6. QuickUpper Hull $\left(p_{\min }, p_{\max }\right)$
7. QuickLowerHull $\left(p_{\min }, p_{\max }\right)$
8. By the previous steps, each node keeps consecutive pieces (if any) of the upper hull as well as the lower hull. However, some nodes can keep empty pieces. In order to obtain a more compact output representation in batches of $n$ consecutive vertices of the hull (but for the last batch) assigned to consecutive nodes of the clique, the nodes can count the number of vertices on the upper and lower hull they hold and send the information to the other nodes. Using the global information, they can design destination addresses for their vertices on both hulls. Then, the routing protocol from Fact 1 can be applied.

## procedure QuickUpperHull $(p, r)$

Input: The upper-hull subsequence of the input point set $S$ held in consecutive nodes in batches of at most $n$ points and two distinguished points $p, r$ in the subsequence, where the $x$-coordinate of $p$ is smaller than that of $r$.
Output: The vertices of the upper hull of $S$ with $x$-coordinates between those of $p$ and $r$ held in clockwise order in consecutive nodes, between those holding $p$ and $r$ respectively, in batches of at most $n$ points.

1. Each node $u$ determines the set $S_{u}$ of points in its upper-hull subsequence that have $x$-coordinates between those of $p$ and $r$ and lie above or on the segment between $p$ and $r$. If $S_{u}$ is not empty then the node sends a point in $S_{u}$ at maximum distance from the line segment between $p$ and $r$ to the clique node holding $p$, from here on referred to as the master node.
2. If the master node has not received any point satisfying the requirements from the previous step then it proclaims $p$ and $r$ to be vertices of the upper hull by sending this information to the nodes holding $p$ and/or $r$, respectively. (In fact one of the vertices $p$ and $r$ has been marked as being on the upper hull earlier.) Next, it pops a call of QuickUpperHull from the top of a stack of recursive calls held in a prefix of the clique nodes numbered $1,2, \ldots$. In case the stack is empty it terminates QuickUpperHull $\left(p_{\text {min }}, p_{\max }\right)$.
3. If the master node has received some points satisfying the requirements from Step 1 then it determines a point $q$ at maximum distance from the line segment between $p$ and $r$ among them; see Fig. 3. Next, it puts the call of QuickUpper $\operatorname{Hull}(q, r)$ on the top of the stack and then activates QuickUpperHull $(p, q)$.

The procedure QuickLower $\operatorname{Hull}(p, r)$ is defined analogously.
Each step of the procedure QuickConvex $\operatorname{Hull}(S)$, but for the calls to QuickUpperHull $\left(p_{\min }, p_{\max }\right)$ and QuickLowerHull $\left(p_{\min }, p_{\max }\right)$, can be done in $O(1)$ rounds on the congested clique on $n$ nodes. In particular, the sorting and the routing steps in QuickConvex $H u l l(S)$ can be done in $O(1)$ rounds by Facts 1, 2. Similarly, each step of QuickUpper $\operatorname{Hull}(p, r)$, and symmetrically each step of QuickLower Hull( $p, r$ ), but for recursive calls, can be done in $O(1)$ rounds. Since each non-leaf (in the recursion tree) call of QuickUpperHull $(p, r)$ and QuickLower $\operatorname{Hull}(p, r)$ results in a new vertex of the convex hull, their total number does not exceed $h$. Hence, we obtain the following theorem.

Theorem 1. Consider a congested n-clique network, where each node holds a batch of $n$ points in the Euclidean plane specified by $O(\log n)$-bit coordinates. Let $h$ be the number of vertices on the convex hull of the set $S$ of the $n^{2}$ points. The convex hull of $S$ can be computed by the procedure QuickConvexHull( $S$ ) in $O(h)$ rounds on the congested clique.

## 4 An $O(\log n)$-round Algorithm for Convex Hull on Congested Clique

Our refined algorithm for the convex hull of the input point set $S$ analogously as QuickConvexHull( $S$ ) described in Section 3 starts by sorting the points in $S$ by their $x$-coordinates and then splitting the sorted sequence of points in $S$ into an upper-hull subsequence and lower-hull subsequence. Next, it computes the upper hull of $S$ and the lower hull of $S$ by calling the procedures NewUpperHull(s) and NewLowerHull(S), respectively. The procedure NewUpperHull(S) lets each node $\ell$ construct the upper hull $H_{\ell}$ of its batch of at most $n$ points in the upperhull subsequence locally. The crucial step of $\operatorname{NewUpperHull}(S)$ is a parallel computation of bridges between all pairs $H_{\ell}, H_{m}, \ell \neq m$, of the constructed upper hulls by parallel calls to the procedure $\operatorname{Bridge}\left(H_{\ell}, H_{m}\right)$. Based on the bridges between $H_{\ell}$ and the other upper hulls $H_{m}$, each node $\ell$ can determine which of the vertices of $H_{\ell}$ belong to the upper hull of $S$ (see Lemma 1). The procedure Bridge has recursion depth $O(\log n)$ and the parallel implementation of the crucial step of $\operatorname{NewUpperHull}(s)$ takes $O(\log n)$ rounds. The procedure NewLowerHull(s) is defined symmetrically. Consequently, the refined algorithm for the convex hull of $S$ specified by the procedure NewConvexHull( $S$ ) can be implemented in $O(\log n)$ rounds.

The procedure NewConvexHull $(S)$ is defined in exactly the same way as QuickConvexHull $(S)$, except that the call QuickUpperHull $\left(p_{\text {min }}, p_{\max }\right)$ in Step

6 is replaced by the call $N e w U p p e r H u l l(S)$ and the call QuickLowerHull $\left(p_{\min }, p_{\max }\right)$ in Step 7 is replaced by the call NewLowerHull $(S)$.

The strategy of $\operatorname{NewUpperHull}(S)$ is to successively identify and mark points in $S$ that cannot belong to the upper hull of $S$ as not qualifying, and finally return the unmarked points.
procedure NewUpperHull(S)
Input: The upper-hull subsequence of the input point set $S$ held in consecutive nodes in batches of at most $n$ points.
Output: The vertices of the upper hull of $S$ held in clockwise order in consecutive nodes in batches of at most $n$ vertices.

1. Each node $\ell$ computes the upper hull $H_{\ell}$ of its upper-hull subsequence locally.
2. In parallel, for each pair $\ell, m$ of nodes, the procedure $\operatorname{Bridge}\left(H_{\ell}, H_{m}\right)$ computing the bridge between $H_{\ell}$ and $H_{m}$ is called. (The procedure uses the two nodes in $O(\log n)$ rounds, exchanging at most two messages between the nodes in each of these rounds.)
3. Each node $\ell$ checks if it has a single point $p$ not marked as not qualifying for the upper hull of $S$ such that there are bridges between $H_{k}$ and $H_{\ell}$ and $H_{\ell}$ and $H_{m}$, where $k<\ell<m, p$ is an endpoint of both bridges, and the angle formed by the two bridges is smaller than 180 degrees. If so, $p$ is also marked as not qualifying for the upper hull of $S$.
4. Each node $\ell$ prunes the set of vertices of $H_{l}$, leaving only those vertices that have not been marked in the previous steps (including calls to the procedure Bridge) as not qualifying for the upper hull of $S$.

The following lemmata enable the implementation of the $n^{2}$ calls to $\operatorname{Bridge}\left(H_{\ell}, H_{m}\right)$ in the second step of NewUpper Hull $(S)$ in $O(\log n)$ rounds on the congested clique.

Lemma 1. For any $\ell \in\{1,2, \ldots, n\}$, let $H_{\ell}$ be the upper hull of the upper-hull subsequence of $S$ assigned to the node $\ell$. A vertex $v$ of $H_{\ell}$ is not a vertex of the upper hull of $S$ if and only if it lies below a bridge between $H_{\ell}$ and $H_{m}$, where $\ell \neq m$, or there are two bridges between $H_{\ell}$ and $H_{k}, H_{m}$, respectively, where $k<\ell<m$, such that they touch $v$ and form an angle of less than 180 degrees at $v$.

Proof. Clearly, if at least one of the two conditions on the right side of "if and only if" is satisfied then $v$ cannot be a vertex of the upper hull of $S$. Suppose that $v$ is not a vertex of the upper hull of $S$. Then, since it is a vertex of $H_{l}$, there must be an edge $e$ of the upper hull of $S$ connecting $H_{k}$ with $H_{m}$ for some $k \leq \ell \leq m, k \neq m$, that lies above $v$. We may assume without loss of generality that $v$ does not lie below any bridge between $H_{\ell}$ and $H_{q}, \ell \neq q$. It follows that $k<\ell<m$. Let $b_{k}$ be the bridge between $H_{k}$ and $H_{\ell}$, and let $b_{m}$ be the bridge between $H_{\ell}$ and $H_{m}$. It also follows that both $b_{k}$ and $b_{m}$ are placed below $e$ and the endpoint of $b_{k}$ at $H_{\ell}$ is $v$ or a vertex of $H_{\ell}$ to the left of $v$ while the endpoint of $b_{m}$ at $H_{\ell}$ is $v$ or a vertex to the right of $v$. Let $C$ be the convex chain that is a
part of $H_{\ell}$ between the endpoints of $b_{k}$ and $b_{m}$ on $H_{\ell}$. Suppose that $C$ includes at least one edge. The bridge $b_{k}$ has to form an angle not less than 180 degrees with the leftmost edge of $C$ and symmetrically the bridge $b_{m}$ has to form an angle not less than 180 degrees with the rightmost edge of $C$. However, this is impossible because the bridges $b_{k}$ and $b_{m}$ are below the edge $e$ of the upper hull of $S$ with endpoints on $H_{k}$ and $H_{m}$ so they form an angle less than 180 degrees. We conclude that $C$ consists solely of $v$ and consequently $v$ is an endpoint of both $b_{k}$ and $b_{m}$. See Fig. 4 .


Fig. 4. The final case in the proof of Lemma 1.

The following folklore lemma follows easily by a standard case analysis (cf. [7|12|13]). See also Fig. 5. It implies that when computing the two endpoints of the bridge between two upper hulls, one can eliminate at least a quarter of all the remaining candidates after looking at six points only. Hence, the recursive depth of the procedure Bridge is $O(\log n)$.


Fig. 5. An example of the segment connecting $m_{1}$ with $m_{2}$ in Lemma 2.

Lemma 2. Let $S_{1}, S_{2}$ be two n-point sets in the Euclidean plane separated by a vertical line. Let $H_{1}, H_{2}$ be the upper hulls of $S_{1}, S_{2}$, respectively. Suppose that
each of $H_{1}$ and $H_{2}$ has at least three vertices. Next, let $m_{1}, m_{2}$ be the median vertices of $H_{1}, H_{2}$, respectively. Suppose that the segment connecting $m_{1}$ with $m_{2}$ is not the bridge between $H_{1}$ and $H_{2}$. Then depending on $m_{1}, m_{2}$ and their neighbors on $H_{1}, H_{2}$, respectively, none of the vertices in at least one of the following four sets is an endpoint of the bridge between $H_{1}$ and $H_{2}$ :
(i) the vertices on $H_{1}$ to the left of $m_{1}$;
(ii) the vertices on $H_{1}$ to the right of $m_{1}$;
(iii) the vertices on $\mathrm{H}_{2}$ to the left of $m_{2}$; and
(iv) the vertices on $\mathrm{H}_{2}$ to the right of $m_{2}$.
procedure Bridge $\left(H_{\ell}^{\prime}, H_{m}^{\prime}\right)$
Input: A continuous fragment $H_{\ell}^{\prime}$ of the upper hull $H_{\ell}$ of the upper-hull subsequence assigned to a node $\ell$ and a continuous fragment $H_{m}^{\prime}$ of the upper hull $H_{m}$ of the upper-hull subsequence assigned to the node $m$.
Output: The bridge between $H_{\ell}^{\prime}$ and $H_{m}^{\prime}$. Moreover, all points in the upper-hull subsequence held in the nodes $\ell$ and $m$ placed under the bridge are marked as not qualifying for the convex hull of $S$.

1. If $H_{\ell}^{\prime}$ or $H_{m}^{\prime}$ has at most two vertices then compute the bridge between $H_{\ell}^{\prime}$ and $H_{m}^{\prime}$ by sending the at most two vertices from $\ell$ to $m$ or vice versa, computing the bridge locally at $m$ or $\ell$, respectively, and sending back the bridge segment from $\ell$ to $m$ or vice versa, respectively. Next, mark all the points in the upper-hull subsequence between the endpoints of the found bridge that are assigned to the nodes $\ell$ or $m$ as not qualifying for vertices of the upper hull of $S$ and stop.
2. Find a median $m_{1}$ of $H_{\ell}^{\prime}$ and a median $m_{2}$ of $H_{m}^{\prime}$.
3. If the straight line passing through $m_{1}$ and $m_{2}$ is a supporting line for both $H_{\ell}^{\prime}$ and $H_{m}^{\prime}$ then mark all the points in the upper-hull subsequence between $m_{1}$ and $m_{2}$ that are assigned to the nodes $\ell$ or $m$ as not qualifying for vertices of the upper hull of $S$ and stop.
4. Otherwise, $\ell$ and $m$ inform each other about the neighbors of $m_{1}$ on $H_{\ell}^{\prime}$ and the neighbors of $m_{2}$ on $H_{m}^{\prime}$, respectively. Then, $\operatorname{Bridge}\left(H_{\ell}^{\prime \prime}, H_{m}^{\prime \prime}\right)$ is called, where either $H_{\ell}^{\prime}=H_{\ell}^{\prime \prime}$ and $H_{m}^{\prime \prime}$ is obtained from $H_{m}^{\prime}$ by removing vertices on the appropriate side of the median of $H_{m}^{\prime}$ or vice versa, according to Lemma 2.

The procedure NewLower $H u l l\left(H_{\ell}^{\prime}, H_{m}^{\prime}\right)$ is defined analogously.
As in the procedure QuickConvexHull $(S)$, each step of NewConvexHull( $S$ ), but for the calls to $\operatorname{NewUpperHull}(S)$ and $\operatorname{NewLowerHull}(S)$, can be done in $O(1)$ rounds on the congested clique by [8]. Furthermore, the first, next to the last, and last steps of $N e w U p p e r H u l l(S)$ require $O(1)$ rounds. By Lemma 2, the recursion depth of the procedure Bridge is logarithmic in $n$. The crucial observation is now that consequently the nodes $\ell$ and $m$ need to exchange $O(\log n)$ messages in order to implement $\operatorname{Bridge}\left(H_{\ell}, H_{m}\right)$. In particular, they need to inform each other about the current medians and their neighbors on $H_{\ell}^{\prime}$ or $H_{m}^{\prime}$, respectively. Also, in case $H_{\ell}^{\prime}$ or $H_{m}^{\prime}$ contains at most two vertices, the node $\ell$ or $m$ needs to inform the other node about the situation and about those at
most two vertices. In consequence, by Lemma 2, these two nodes can implement $\operatorname{Bridge}\left(H_{\ell}, H_{m}\right)$ by sending a single message to each other in each round in a sequence of $O(\log n)$ consecutive rounds. It follows that all the $n^{2}$ calls of Bridge $\left(H_{\ell}, H_{m}\right)$ can be implemented in parallel in $O(\log n)$ rounds. Note that in each of the $O(\log n)$ rounds, each clique node sends at most one message to each other clique node, so in total, each node sends at most $n-1$ messages to the other nodes in each of these rounds. It follows that NewUpperHull $(S)$ and symmetrically $\operatorname{NewLowerHull}(S)$ can be implemented in $O(\log n)$ rounds on the congested clique. We conclude that NewConvexHull( $S$ ) can be done in $O(\log n)$ rounds on the congested clique.

Theorem 2. Consider a congested n-clique network, where each node holds a batch of $n$ points in the Euclidean plane specified by $O(\log n)$-bit coordinates. The convex hull of the set $S$ of the $n^{2}$ input points can be computed by the procedure NewConvexHull $(S)$ in $O(\log n)$ rounds on the congested clique.

## 5 Point Set Triangulation in $O\left(\log ^{2} n\right)$ Rounds on Congested Clique

Our method of triangulating a set of $n^{2}$ points in the congested $n$-clique model initially resembles that of constructing the convex hull of the points. That is, first the input point set is sorted by $x$-coordinates. Then, each node triangulates its sorted batch of $n$ points locally. Next, the triangulations are pairwise merged and extended to triangulations of doubled point sets by using the procedure Merge in parallel in $O(\log n)$ phases. In the general case, the procedure Merge calls the procedure Triangulate in order to triangulate the area between the sides of the convex hulls of the two input triangulations, facing each other.

The main idea of the procedure Triangulate is to pick a median vertex $v$ on the longer of the convex hulls sides and send its coordinates and the coordinates of its neighbors to the nodes holding the facing side of the other hull. The latter nodes send back candidates (if any) for a mate $u$ of the median vertex $v$ such that the segment between $v$ and $u$ can be an edge of a triangulation extending the existing partial triangulation. The segment is used to split the area to triangulate into two that are triangulated by two recursive calls of Triangulate in parallel. See Fig. 6. Before the recursive calls the edges of the two polygons surrounding the two areas are moved to new node destinations so each of the polygons is held by a sequence of consecutive clique nodes. This is done by a global routing in $O(1)$ rounds serving all parallel calls of Triangulate on a given recursion level, for a given phase of Merge (its first argument).

Since the recursion depth Triangulate is $O(\log n)$ and Merge is run in $O(\log n)$ phases, the total number of required rounds becomes $O\left(\log ^{2} n\right)$.

To simplify the presentation, we shall assume that the size $n$ of the clique network is a power of 2 .

## procedure Triangulation $(S)$

1. Sort the points in $S$ by their $x$-coordinates so each node receives a subsequence consisting of $n$ consecutive points in $S$, in the sorted order.
2. Each node sends the first point and the last point in its subsequence to the other nodes.
3. Each node $q$ constructs a triangulation $T_{q, q}$ of the points in its sorted subsequence locally.
4. For $1 \leq p<q \leq n, T_{p, q}$ will denote the already computed triangulation of the points in the sorted subsequence held in the nodes $p$ through $q$. For $i=0, \ldots, \log n-1$, in parallel, for $j=1,1+2^{i+1}, 1+2 \cdot 2^{i+1}, 1+3 \cdot 2^{i+1}, \ldots$ the union of the triangulations $T_{j, j+2^{i}-1}$ and $T_{j+2^{i}, j+2^{i+1}-1}$ is transformed to a triangulation $T_{j, j+2^{i+1}-1}$ of the sorted subsequence held in the nodes $j$ through $j+2^{i+1}-1$ by calling the procedure $\operatorname{Merge}(i, j)$.
procedure $\operatorname{Merge}(i, j)$
Input: A triangulation $T_{j, j+2^{i}-1}$ of the subsequence held in the nodes $j$ through $j+2^{i}-1$ and a triangulation $T_{j+2^{i}, j+2^{i+1}-1}$ of the subsequence held in the nodes $j+2^{i}$ through $j+2^{i+1}-1$, .
Output: A triangulation $T_{j, j+2^{i+1}-1}$ of the subsequence held in the nodes $j$ through $j+2^{j+1}-1$.
5. Compute the bridges between the convex hulls of $T_{j, j+2^{i}-1}$ and $T_{j+2^{i}, j+2^{i+1}-1}$. Determine the polygon $P$ formed by the bridges between the convex hulls of $T_{j, j+2^{i}-1}$ and $T_{j+2^{i}, j+2^{i+1}-1}$, the right side of the convex hull of $T_{j, j+2^{i}-1}$, and the left side of the convex hull of $T_{j+2^{i}, j+2^{i+1}-1}$ between the bridges.
6. Triangulate $\left(P, j, j+2^{i+1}-1\right)$


Fig. 6. An example of the partition of the polygon $P$ into the subpolygons $P_{1}, P_{2}$ in the procedure Triangulate.
procedure Triangulate $(P, p, q)$
Input: A simple polygon $P$ composed of two convex chains facing each other on opposite sides of a vertical line and two edges crossing the line, held in nodes $p$ through $q$, with $p<q$.
Output: A triangulation of $P$ held in nodes $p$ through $q$.

1. If $p=q$ then the $p$ node triangulates $P$ locally and terminates the call of the procedure.
2. The nodes $p$ through $q$ determine the lengths of the convex chains on the border of $P$ and the node holding the median vertex $v$ of the longest chain (in case of ties, the left chain) sends the coordinates of $v$ and the adjacent vertices on the chain to the other nodes $p$ through $q$.
3. The nodes holding vertices of the convex chain that is opposite to the convex chain containing $v$ determine if they hold vertices $u$ that could be connected by a segment with $v$ within $P$. They verify if the segment $(v, u)$ is within the intersection of the union of the half-planes on the side of $P$ induced by the edges adjacent to $v$ with the union of the half-planes on the side of $P$ induced by the edges adjacent to $u$. If so, they send one such a candidate vertex $u$ to the node holding $v$.
4. The node holding $v$ selects one of the received candidate vertices $u$ as the mate and sends its coordinates to the other nodes $p$ through $q$.
5. The nodes $p$ through $q$ split the polygon $P$ into two subpolygons $P_{1}$ and $P_{2}$ by the edge $(v, u)$ and by exchanging messages in $O(1)$ rounds compute the new destinations for the edges of the polygons $P_{1}$ and $P_{2}$ so $P_{1}$ can be held in nodes $p$ through $r_{1}$ and $P_{2}$ in the nodes $r_{2}$ through $q$, where $p \leq r_{1} \leq r_{2} \leq q$ and $r_{1}=r_{2}$ or $r_{2}=r_{1}+1$.
6. A synchronized global routing in $O(1)$ rounds corresponding to the current phase of the calls to the procedure Merge (given by its first argument) and all parallel calls of the procedure Triangulate on the same recursion level is implemented by using Fact 1. In particular, the edges of $P_{1}$ and $P_{2}$ are moved to the new consecutive destinations among nodes $p$ through $q$.
7. In parallel, Triangulate $\left(P_{1}, p, r_{1}\right)$ and Triangulate $\left(P_{2}, r_{2}, q\right)$ are performed.

At the beginning, we have outlined our triangulation method, in particular the procedures forming it, in a top-down fashion. We now complement this outline with a bottom-up analysis. All steps of the procedure Triangulate $(P, p, q)$ but for the recursive calls in the last step and the next to the last step can be implemented in $O(1)$ rounds, using only the nodes $p$ through $q$. The next to the last step is a part of the global routing. It serves all calls of the procedure Triangulate on the same recursion level for a given phase of the parallel calls of procedure $\operatorname{Merge}(i$,$) , i.e., for given i$. Since each node is involved in at most two of the aforementioned calls of Triangulate that cannot be handled locally, the global routing, implementing the next to the last step of Triangulate, requires $O(1)$ rounds. Since the recursion depth of Triangulate is $O(\log n)$, Triangulate takes $O(\log n)$ rounds. The first step of the procedure $\operatorname{Merge}(i, j)$, i.e., constructing the bridges between the convex hulls, can be implemented in $O(\log n)$ rounds by using the convex hull algorithm from Section 4 on nodes $j$ through $j+2^{i+1}-1$. The second step can easily be implemented in $O(1)$ rounds using the aforementioned nodes. Finally, the call to Triangulate in the last step of Merge requires $O(\log n)$ rounds by our analysis of this procedure. Again, it can be done by nodes $j$ through $j+2^{i+1}-1$ but for the last steps of calls to Triangulate that are served by the discussed synchronized global routing in $O(1)$ rounds. We
conclude that $\operatorname{Merge}(i, j)$ can be implemented in $O(\log n)$ rounds. Finally, all steps in Triangulation $(S)$ except the one involving parallel calls to $\operatorname{Merge}(i, j)$ in $O(\log n)$ phases can be done in $O(1)$ rounds. For a given phase, i.e., given $i$, each node is involved in $O(1)$ calls of $\operatorname{Merge}(i, j)$ but for the next to the last steps in Triangulate that for a given recursion level of Triangulate are implemented by the joint global routing in $O(1)$ rounds. It follows from our analysis of $\operatorname{Merge}(i, j)$ and $i=O(\log n)$ that Triangulate $(S)$ can be implemented in $O\left(\log ^{2} n\right)$ rounds.

Theorem 3. Consider a congested n-clique network, where each node holds a batch of $n$ points in the Euclidean plane specified by $O(\log n)$-bit coordinates. A triangulation of the set $S$ of the $n^{2}$ input points can be computed by the procedure Triangulation $(S)$ in $O\left(\log ^{2} n\right)$ rounds on the congested clique.

## 6 On the Construction of Voronoi Diagram on Congested Clique

The primary difficulty in the design of efficient parallel algorithms for the Voronoi diagram of a planar point set using a divide-and-conquer approach is the efficient parallel merging of Voronoi diagrams. In [1], Aggarwal et al. presented a very involved $O(\log n)$-time PRAM method for the parallel merging. As a result, they obtained an $O\left(\log ^{2} n\right)$-time CREW PRAM algorithm for the Voronoi diagram. Their work and later improved PRAM algorithms for the Voronoi diagram [416] suggest that this problem should be solvable in $(\log n)^{O(1)}$ rounds on the congested clique.

When the points with $O(\log n)$-bit coordinates are drawn uniformly at random from a unit square or circle then the expected number of required rounds to compute the Voronoi diagram or the dual Delaunay triangulation on the congested clique becomes $O(1)$ (cf. [9|16]). To demonstrate this we need to recall the Chernoff bounds.

Fact 3 (multiplicative Chernoff lower bound) Suppose $X_{1}, \ldots, X_{n}$ are independent random variables taking values in $\{0,1\}$. Let $X$ denote their sum and let $\mu=E[X]$ denote the sum's expected value. Then, for any $\delta \in[0,1], \operatorname{Prob}(X \leq$ $(1-\delta) \mu) \leq e^{-\frac{\delta^{2} \mu}{2}}$ holds. Similarly, for any $\delta \geq 0$, $\operatorname{Prob}(X \geq(1+\delta) \mu) \leq e^{-\frac{\delta^{2} \mu}{2+\delta}}$ holds.

We shall say that an event dependent on $n^{2}$ input points in the plane holds with high probability (w.h.p.) if its probability is at least $1-\frac{1}{n^{\alpha}}$ asymptotically, (i.e., there is an integer $n_{0}$ such that for all $n \geq n_{0}$, the probability is at least $1-\frac{1}{n^{\alpha}}$ ), where $\alpha$ is a constant not less than 2 .

Theorem 4. The Voronoi diagram of $n^{2}$ points with $O(\log n)$-bit coordinates drawn uniformly at random from a unit square in the Euclidean plane can be computed within the square w.h.p. in $O(1)$ rounds on the congested clique.

Proof. Consider an arbitrary square $R$ of size $\frac{1}{\sqrt{n}} \times \frac{1}{\sqrt{n}}$ within the unit square. Next, consider a sequence of $n^{2}$ uniform random draws of points with $O(\log n)$ bit coordinates from the unit square. Call a draw in the sequence a success if a point within $R$ is drawn. The expected number of successes is $n$. Hence, it follows by selecting $\delta=\sqrt{\frac{6 \ln n}{n}}$ in the Chernoff bounds that $\Theta(n)$ of the drawn points are within $R$ with probability at least $1-\frac{1}{n^{3}}$.

Partition the unit square into $n$ rectilinear squares of size $\frac{1}{\sqrt{n}} \times \frac{1}{\sqrt{n}}$.


Fig. 7. An example of the configuration in the proof of Theorem 4

Let $S$ be the set of $n^{2}$ points drawn from the unit square. Consider the Voronoi diagram of $S$ within the unit square. Let $e$ be an edge of the Voronoi diagram. The edge $e$ has to be a part of the bisector of some couple of points $s_{1}$ and $s_{2}$ in $S$. Consider an arbitrary point $q$ on $e$ and the rectilinear square $Q$ in the aforementioned partition that contains it. Suppose that $s_{1}$ or $s_{2}$ lies outside the rectilinear area formed by $Q$ and the two layers of squares around $Q$ in the partition, i.e., consisting of at most $1+8+16=25$ squares including $Q$. See Fig. 7. Without loss of generality, let $s_{2}$ be such a point. Then the distance between $q$ and $s_{2}$ is at least $2 \cdot \frac{1}{\sqrt{n}}$, while the distance between $q$ and every point inside $Q$ is at most $\sqrt{2} \cdot \frac{1}{\sqrt{n}}$. We obtain a contradiction w.h.p. because $Q$ contains $\Omega(n)$ points from $S$ w.h.p. and $q$ is closer to each of these points than to $s_{2}$. It follows that to compute the Voronoi diagram of $S$ within a square in the partition w.h.p. one needs solely to know the points in $S$ located in $Q$ and the at most 24 squares around the square.

We can assign to each of the squares in the partition a distinct clique node and deliver to each node the points from its square in $O(1)$ rounds w.h.p. by using the sorting and routing $O(1)$-round algorithms from Facts 1, 2. Then, additionally we need to deliver to each node the points in $S$ located in the at most 24 squares around its square. By using again the routing algorithms from

Facts 1, 2, this can be achieved in $O(1)$ rounds w.h.p. (Note that the total number of points that need to be delivered to each node is $O(n)$ w.h.p. since each of the squares contains $O(n)$ points w.h.p.) Finally, each node applies any sequential Voronoi diagram algorithm (e.g., [6]) to locally compute the Voronoi diagram of the w.h.p. $O(n)$ many points it received and then it determines its intersection with the square assigned to the node. This shows that each node computes its local Voronoi diagram correctly w.h.p., and according to the first paragraph above, this probability is at least $1-\frac{1}{n^{3}}$. Now, the union bound implies that the probability that all nodes compute their local Voronoi diagrams correctly is at least $1-\frac{1}{n^{2}}$.

## 7 Concluding Remarks

We have provided the first non-trivial, polylogarithmic upper bounds on the number of rounds required to construct the convex hull and a triangulation of a set of $n^{2}$ points in the plane with $O(\log n)$-bit coordinates in the model of congested clique. As for the construction of the Voronoi diagram of the point set, we have shown an $O(1)$ upper bound on the number of rounds under the assumption that the points are drawn uniformly at random from a unit square. The major open problem is the derivation of a non-trivial upper bound on the number of rounds sufficient to construct the Voronoi diagram when the points are not necessarily randomly distributed. This seems to be possible but it might require a substantial effort; see the discussion in the preceding section. An interesting question is also if a simple polygon on $n^{2}$ vertices with $O(\log n)$-bit coordinates can be triangulated using a substantially smaller number of rounds than that needed to triangulate a set of $n^{2}$ points in the plane with $O(\log n)$ bit coordinates in the congested $n$-clique model.

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