Graph Orientation to Maximize the Minimum Weighted Outdegree

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Abstract

We study a new variant of the graph orientation problem called MAXMINO *where the input is an undirected, edge-weighted graph and the objective is to assign a direction to each edge so that the minimum weighted outdegree (taken over all vertices in the resulting directed graph) is maximized. All edge weights are assumed to be positive integers. This problem is closely related to the job scheduling on parallel machines, called the machine covering problem, where its goal is to assign jobs to parallel machines such that each machine is covered as much as possible. First, we prove that* MAXMINO *is strongly NP-hard and cannot be approximated within a ratio of $2-\epsilon$ for any constant $\epsilon > 0$ in polynomial time unless P=NP, even if all edge weights belong to $\{1,2\}$, every vertex has degree at most three, and the input graph is bipartite or planar. Next, we show how to solve* MAXMINO *exactly in polynomial time for the special case in which all edge weights are equal to* 1*. This technique gives us a simple polynomial-time $\frac{w_{max}}{w_{min}}$-approximation algorithm for* MAXMINO *where w_{max} and w_{min} denote the maximum and minimum weights among all the input edges. Furthermore, we also observe that this approach yields an exact algorithm for the general case of* MAXMINO *whose running time is polynomial whenever the number of edges having weight larger than w_{min} is at most logarithmic in the number of vertices. Finally, we show that* MAXMINO *is solvable in polynomial time if the input is a cactus graph.*

Keywords: Graph orientation, Approximation algorithm, Hardness of approximation.

1. Introduction

An orientation of an undirected graph is an assignment of a direction to each of its edges. Graph orientation is a well-studied area of graph theory and combinatorial optimization and thus a large variety of objective functions have been considered so far. The objective function of the present paper is the maximization of the minimum outdegree. It is closely related to the classic job scheduling on parallel machines. In the parallel machine scheduling scenario, our problem can be regarded as the restricted assignment variant of the machine covering problem [18], where its goal is to assign jobs to parallel machines such that each machine is covered as much as possible. In the following, we first define several terminologies and our objective function, then describe related work, and summarize our results.

Problem definition. Let $G = (V, E, w)$ be a given undirected, edge-weighted graph with vertex set V and edge set E whose weights are numbers specified by a function w. An *orientation* Λ of G is defined to be any function on E of the form $\Lambda : \{u, v\} \mapsto \{(u, v), (v, u)\}$, i.e., an assignment of a direction to each undirected edge $\{u, v\}$ in E. Given an orientation Λ of G, the *weighted outdegree* $d_\Lambda(v)$ of a vertex $v \in V$ is defined as the total weight of all edges leaving v, i.e., $d_\Lambda(v) = \sum_{\substack{\{u,v\}\in E:\\ \Lambda(\{u,v\})=(v,u)}} w(\{u, v\})$, and the *minimum weighted outdegree* $\delta_\Lambda(G)$ is defined by

$\delta_\Lambda(G) = \min_{v \in V}\{d_\Lambda(v)\}$.

In this paper we deal with the problem of finding an orientation of the input graph such that the minimum weighted outdegree is maximum. We call this problem Maximum Minimum Weighted Outdegree Graph Orientation Problem (MAXMINO for short): The input is an undirected, edge-weighted graph $G = (V, E, w)$ with $w : E \to \mathbb{Z}^+$, where $\mathbb{Z}^+$ denotes the set of positive integers, and the objective is to find an orientation Λ^* of G which maximizes $\delta_\Lambda(G)$ over all possible orientations Λ of G. Such an orientation is called a *max-min orientation of* G, and the corresponding value $\delta_{\Lambda^*}(G)$ is denoted by $OPT(G)$. The special case of MAXMINO where all edge weights of the input graph are equal to 1 is referred to as *unweighted* MAXMINO.

Throughout the paper, we use the following notations: $n = |V|$, $m = |E|$, and $W = \sum_{e \in E} w(e)$ for the input G. Furthermore, w_{max} and w_{min} denote the maximum and minimum weights, respectively, among all edges in E. For any $v \in V$, the (unoriented) *weighted degree of* v, denoted by $d(v)$, is the sum of all weights of edges incident to v, and $\Delta = \max_{v \in V}\{d(v)\}$ is the maximum (unoriented) weighted degree among all vertices in G. Also for a (fixed) $v \in V$, we call $|\{\{u, v\} \in E\}|$ (i.e., the number of edges incident to v) the (unoriented) *unweighted degree of* v, and denote it by $deg(v)$. We also call $\max_{v \in V} deg(v)$ the (unoriented) *unweighted degree of* G, both of which will be used to focus on the topological structure of the graph.

We say that an algorithm $\mathcal{A}$ is a σ*-approximation algorithm* for MAXMINO or that $\mathcal{A}$'s *approximation ratio* is at most σ, if $OPT(G) \leq \sigma \cdot \mathcal{A}(G)$ holds for any input graph G, where $\mathcal{A}(G)$ is the minimum weighted outdegree in the orientation returned by $\mathcal{A}$ on input G.

Related work. MAXMINO studied in the current work is closely related to the restricted assignment variant of the machine covering problem, which is often called the Santa Claus problem [4], [5], [8], [12]: Santa Claus has m gifts (corresponding to jobs, and to edges in MAXMINO) that he wants to distribute among n kids (corresponding to machines, and to vertices in MAXMINO). Some gift may be worth \$100 but another may be not so expensive, and some kids do not want some of the gifts whatsoever (i.e., its value is 0 for the kids). The goal of Santa Claus is to distribute the gifts in such a way that the least lucky kid is as happy as possible. In addition, MAXMINO has the following restriction (which might be strong and somehow strange in the Santa Claus scenario): Every gift is of great value only to exactly two kids and thus it must be delivered to one of them. For the Santa Claus problem, Golovin [12] provided an $O(\sqrt{n})$-approximation algorithm for the restricted case where the value of each gift belongs to $\{1, k\}$ for some integer k. Bansal and Sviridenko [4] considered the general value case and showed that a certain linear programming relaxation can be used to design an $O(\log\log m / \log\log\log m)$-approximation algorithm, while Bezakova and Dani [5] already showed that the general case is NP-hard to approximate within ratios smaller than 2.

Another objective function studied for the graph orientation problem is that of *minimizing the maximum weighted outdegree* (MINMAXO), also known as *Graph Balancing* [1], [2], [3], [7], [13], [17]: Given an undirected graph with edge weights, we are asked to assign a direction to each edge so that the maximum outdegree is minimized. It is obvious that MINMAXO is generally NP-hard. Asahiro et al. [2] showed that it is still weakly NP-hard for outerplanar graphs, and strongly NP-hard for P_4-bipartite graphs. Fortunately, however, they also showed [2] that MINMAXO is tractable if the input is limited to trees or even to cactus graphs. Note that the class of cactus graphs is a maximal subset of the class of outerplanar graphs and the class of P_4-bipartite graphs, and a minimal superset of the class of trees. Very recently, Ebenlendr et al. [7] designed a polynomial-time 1.75-approximation algorithm for the general weighted case, and Asahiro et al. [1] showed that MINMAXO can be approximated within an approximation ratio of 1.5 in polynomial-time if all edge weights belong to $\{1, 2\}$. As for inapproximability, it is known that MINMAXO is NP-hard to approximate within approximation ratios smaller than 1.5 even for this restricted $\{1, 2\}$-case [1], [7].

Our results. In this paper we study the computational complexity and (in)approximability of the machine covering problem from the viewpoint of the graph based problem, i.e., graph orientation. In Section 2, we prove that MAXMINO is strongly NP-hard and cannot be approximated within a ratio of $\min\{2, \frac{w_{max}}{w_{min}}\} - \epsilon$ for any constant $\epsilon > 0$ in polynomial time unless P=NP, even if all edge weights belong to $\{w_{min}, w_{max}\}$, every vertex has unweighted degree at most three, and the input graph is bipartite and planar. As mentioned above, although MAXMINO imposes a strong restriction on the Santa Claus problem, unfortunately it is still hard.

Section 3 first considers the unweighted MAXMINO problem. We can obtain an optimal orientation algorithm which runs in $O(m^{3/2} \cdot \log m \cdot \log^2 \Delta)$ time for the special case in which all edge weights are equal to 1. Here, it is important to note that Golovin [12] already claimed that the unweighted case of MAXMINO (more precisely, the Santa Claus problem) can be solved in polynomial time, but no proof of this claim has ever appeared as far as the authors know. Our contribution here is to provide the non-trivial, efficient running time with its explicit proof. Then, we observe that our approach yields an exact algorithm for the general case of MAXMINO whose running time is polynomial whenever the number of edges having weight larger than w_{min} is at most logarithmic in the number of vertices. In Section 4, this efficient algorithm for the unweighted MAXMINO also gives us a simple $\frac{w_{max}}{w_{min}}$-approximation algorithm running in the same time for general (weighted) case of MAXMINO, i.e., it always outputs an orientation Λ' of G which satisfies $OPT(G) \leq \frac{w_{max}}{w_{min}} \cdot \delta_{\Lambda'}(G)$. This simple approximation algorithm is best possible for the case that the weights of edges belong to $\{w_{min}, w_{max}\}$ with $w_{max} \leq 2w_{min}$ since the lower bound of approximation ratios is $\min\{2, \frac{w_{max}}{w_{min}}\}$ described above.

In the field of combinatorial optimization, much work is often devoted to seek a subset of instances that is tractable and as large as possible. For example, if the input graph G is a tree, then $OPT(G)$ is always 0 because the number of vertices is larger than the number of edges, and in any orientation of G, at least one vertex must have no outgoing edges. Also, for the case of cycles, MAXMINO is quite trivial since the clockwise or counterclockwise orientation along the cycle gives us the optimal value of w_{min}. On the other hand, the class of planar graphs is too large to allow a polynomial-time optimal algorithm (under the assumption of P≠NP). Hence, our goal in Section 5 is to find a polynomially solvable subset between trees and planar graphs. Then, we show that MAXMINO remains in P even if we make the set of instances so large that it contains the class of cactus graphs.

2. Hardness results

In this section, we show the MAXMINO problem is strongly NP-hard even if all the edge weights belong to $\{w_{min}, w_{max}\}$ for any integers $w_{min} < w_{max}$ and the input graph is bipartite and planar. The proof is by a reduction from AT-MOST-3-SAT(2L).

AT-MOST-3-SAT(2L) is a restriction of 3-SAT where each clause contains at most three literals and each literal (not variable) appears at most twice in a formula. It can be easily proved that AT-MOST-3-SAT(2L) is NP-hard by using problem [LO1] on p. 259 of [9].

First, we pick any fixed integers for w_{min} and w_{max} such that $w_{min} < w_{max}$. Given a formula ϕ of AT-MOST-3-SAT(2L) with n variables $\{v_1, \ldots, v_n\}$ and m clauses $\{c_1, \ldots, c_m\}$, we then construct a graph G_ϕ including gadgets that mimic (a) variables and (b) clauses. To define these, we prepare a gadget consisting of a cycle of 3 vertices and 3 edges (i.e., a triangle) where each edge of the cycle has weight w_{max}. We call this a *triangle gadget*. Apart from these triangle gadgets, we define gadgets for (a) variables and (b) clauses: (a) Each variable gadget corresponding to a variable v_i consists of two vertices labeled by v_i and $\overline{v_i}$ and one edge $\{v_i, \overline{v_i}\}$ between them. The weight of $\{v_i, \overline{v_i}\}$ is w_{max}. By the definition of AT-MOST-3-SAT(2L), some literals (say v_i for example) do not occur (or may occur only once). In such a case, we attach a triangle gadget to the variable gadget by adding two edges (one edge) of weight w_{min} that connects vertex v_i and two different vertices (one vertex) of the triangle gadget. (b) Each clause gadget consists of one representative vertex labeled by c_j, corresponding to clause c_j of ϕ, and a triangle gadget connected to this c_j-vertex by an edge of weight w_{min}. The representative vertex c_j is also connected to at most three vertices in the literal gadgets that have the same labels as the literals in the clause c_j, by edges of weight w_{min}. For example, if $c_1 = x \vee \overline{y}$ appears in ϕ, then vertex c_1 is connected to vertices x and $\overline{y}$. (See Figure 1.) We have the following lemma.

Lemma 1: For the reduced graph G_ϕ, the following holds:

(i) $OPT(G_\phi) \geq \min\{2w_{min}, w_{max}\}$ if ϕ is satisfiable.

(ii) $OPT(G_\phi) \leq w_{min}$ if ϕ is not satisfiable.

Proof: First, if $\delta_\Lambda(G) > w_{min}$ for an optimal orientation Λ, then we can assume that each triangle gadget is oriented in such a way that the triangle forms a directed cycle in an optimal orientation, which guarantees that the minimum weighted outdegree among those vertices belonging to the triangle is at least w_{max} (otherwise, it has a vertex whose weighted outdegree is at most w_{min}). Due to this cycle orientation, we can also assume that edges that connect triangle gadgets to other vertices are oriented towards the triangle gadgets in the optimal orientation.

Now we prove (i). Suppose that there is a satisfying truth assignment τ for the formula ϕ. From τ, we construct an orientation with $OPT(G_\phi) \geq \min\{2w_{min}, w_{max}\}$. If $v_i = true$ in τ, the edge $\{v_i, \overline{v_i}\}$ is oriented from v_i to $\overline{v_i}$; otherwise, from $\overline{v_i}$ to v_i. At this moment, the weighted outdegree of vertices associated with the literals of *true* and *false* assignments is w_{max} and 0, respectively. (We call the vertices associated with literals of *true* (resp., *false*) assignments *true* (resp., *false*) vertices. For example, if a variable $x = false$ in a truth assignment, then the upper leftmost vertex x is called a false vertex and the second leftmost vertex $\overline{x}$ is called a true vertex in Figure 1.) Each false vertex has one or two edges connected to clause vertices, and in case a false vertex is connected to one clause vertex, then it is connected to a triangle gadget. We then orient such edges towards the clause vertices and triangle gadgets, which make the weighted outdegree of each false vertex $2w_{min}$. Thus the weighted outdegree of each vertex in a variable gadget is at least $2w_{min}$. Each clause vertex has at least one edge connected to a true vertex due to the truth assignment. We orient this edge towards the true vertex, which makes the weighted outdegree of the clause vertex at least $2w_{min}$, because it has an edge connected to a triangle gadget. Hence, the weighted outdegree of every vertex is at least $\min\{2w_{min}, w_{max}\}$, which shows (i).

Next, we prove (ii) by showing that if the graph G_ϕ has an orientation whose minimum weighted outdegree is at least $\min\{2w_{min}, w_{max}\}$, then ϕ is satisfiable by constructing the satisfying truth assignment. If an edge in the ith variable gadget v_i is oriented from v_i to $\overline{v_i}$, then we assign $v_i = true$; otherwise, $v_i = false$. Then two edges between a vertex assigned with *false* and its two adjacent

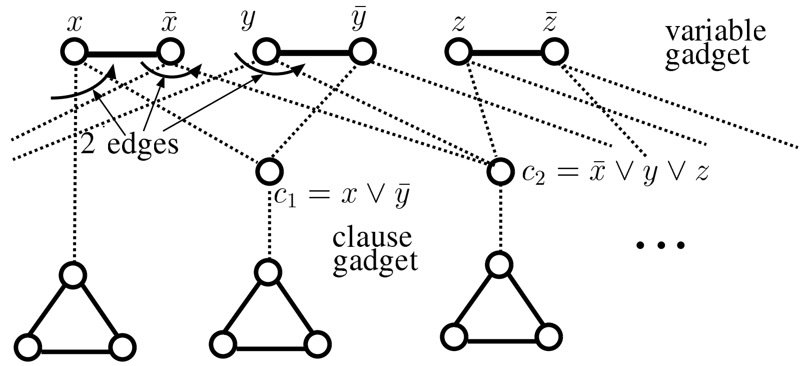

Figure 1. Reduction from AT-MOST-3-SAT(2L) (Solid and dotted edges have weight w_{max} and w_{min}, respectively.)

clause vertices must be oriented towards the clause vertices. (Otherwise, the weighted outdegree of the false vertex is at most w_{min}, which contradicts the assumption.) Every clause vertex is connected with the variable and triangle gadgets. As mentioned above, every edge between a clause vertex and its triangle gadget can be assumed to be oriented towards the triangle gadget. It follows that for each clause vertex, there must be at least one edge directed towards the clause vertex from a vertex v in a variable gadget, and v must be a true vertex. This means that the above truth assignment satisfies all the clauses in ϕ. □

From Lemma 1, we immediately obtain the following theorem.

Theorem 2: MAXMINO is strongly NP-hard even if the edge weights are in $\{w_{min}, w_{max}\}$ ($w_{min} < w_{max}$). □

Also the (un)satisfiability gap of Lemma 1 yields the following theorem.

Theorem 3: Even if the edge weights are in $\{w_{min}, w_{max}\}$, MAXMINO has no pseudo-polynomial time algorithm whose approximation ratio is smaller than $\min\{2, \frac{w_{max}}{w_{min}}\}$, unless P = NP. □

Similarly, we can show the NP-hardness of MAXMINO for planar bipartite graphs by almost the same reduction as the above from MONOTONE-PLANAR-ONE-IN-THREE-3-SAT(2L), which is a variant of AT-MOST-3-SAT(2L), having both the planarity [14] and the monotonicity [10].

ONE-IN-THREE-3-SAT itself is a variant of 3-SAT problem which asks whether there exists a truth assignment to the variables so that each clause has exactly one true literal (and thus exactly two false literals) [16]. The reason why we use ONE-IN-THREE-3-SAT instead of AT-MOST-3-SAT is to bound the unweighted degrees of the constructed graphs. While the above reduction from AT-MOST-3-SAT(2L) guarantees that the unweighted degrees of constructed graphs are bounded by four, we can bound the unweighted degrees of constructed graph from ONE-IN-THREE-3SAT(2L) by three. In the new reduction, we do not attach triangle gadgets to clause vertices, which makes the unweighted degrees of clause vertices three, and One-In-Three satisfiability guarantees that each clause vertex has two outgoing edges in an optimal MAXMINO solution.

The *planarity* means that the graph constructed from an instance CNF, in which two vertices corresponding to a variable and a clause are connected by an edge if the variable occurs (positively or negatively) in the clause, is planar. The *monotonicity* means that in an input CNF formula each clause contains either only positive literals or only negative literals. PLANAR-ONE-IN-THREE-3-SAT is shown to be NP-complete in [15].

By applying an operation used in [2], we can transform an instance of PLANAR-ONE-IN-THREE-3-SAT into one of MONOTONE-PLANAR-ONE-IN-THREE-3-SAT. Moreover, by applying another operation used in the same paper [2], we can transform an instance of MONOTONE-PLANAR-ONE-IN-THREE-3-SAT into MONOTONE-PLANAR-ONE-IN-THREE-3-SAT(2L). This implies that the constructed graph is planar and bipartite and its unweighted degree is at most three. (To preserve the bipartiteness, we need to use bipartite gadgets, e.g., square gadgets, instead of triangle gadgets.)

Theorem 4: MAXMINO is strongly NP-hard even if the edge weights are in $\{w_{min}, w_{max}\}$ for integers $w_{min} < w_{max}$ and the input graph is bipartite and planar in which the unweighted degree is bounded by three. □

Theorem 5: Even if the edge weights are in $\{w_{min}, w_{max}\}$ and the input graph is bipartite and planar in which the unweighted degree is bounded by three, MAXMINO has no pseudo-polynomial time algorithm whose approximation ratio is smaller than $\min\{2, \frac{w_{max}}{w_{min}}\}$, unless P=NP. □

This result is tight in a sense, because if the unweighted degree of the input graph is bounded by two (i.e., cycles or trees), obviously MAXMINO can be solved in linear time.

3. An exact algorithm for unweighted cases

MAXMINO is closely related to the problem of computing a maximum flow in a flow network with positive edge capacities. Indeed, maximum-flow-based techniques have been used in [3] to solve the analogous problem of computing an

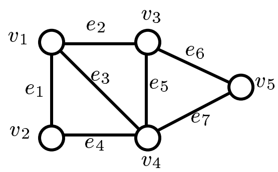

Figure 2. An example of G

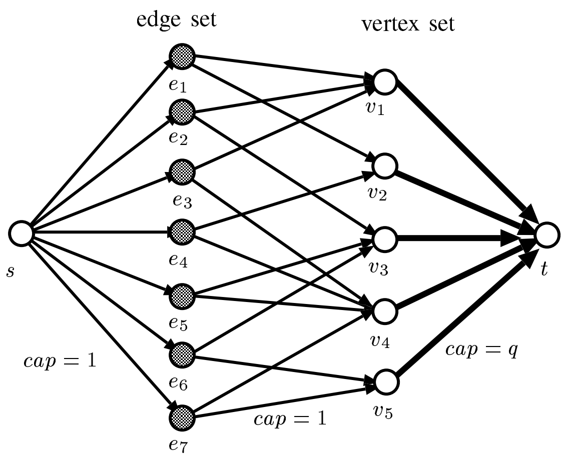

Figure 3. Network $\mathcal{N}_G$ constructed from G of Figure 2

edge orientation which *minimizes* the *maximum* outdegree of a given unweighted graph (MINMAXO) in polynomial time. In this section, we extend the results of [3] by showing how a maximum flow-algorithm can be used to efficiently solve unweighted MAXMINO.

For any input graph $G = (V, E)$ to unweighted MAXMINO, let $\mathcal{N}_G = (V_G, E_G)$ be the directed graph with vertex set V_G and edge set E_G defined by:

$$\begin{aligned} V_G &= E \cup V \cup \{s, t\}, \\ E_G &= \big\{(s,e) \mid e \in E\big\} \cup \big\{(v,t) \mid v \in V\big\} \cup \\ &\quad \big\{(e, v_i), (e, v_j) \mid e = \{v_i, v_j\} \in E\big\}, \end{aligned}$$

and for any integer $q \in \{0, 1, \ldots, \Delta\}$, let $\mathcal{N}_G(q) = (V_G, E_G, cap_q)$ be the flow network obtained by augmenting $\mathcal{N}_G$ with edge capacities cap_q, where:

$$cap_q(a) = \begin{cases} 1, & \text{if } a = (s, e) \text{ with } e \in E; \\ 1, & \text{if } a = (e, v) \text{ with } e \in E,\ v \in V; \\ q, & \text{if } a = (v, t) \text{ with } v \in V. \end{cases}$$

See Figure 2 and Figure 3 for an example of the original graph G and the corresponding network $\mathcal{N}_G$, respectively.

Let $F(q)$ be an integral maximum directed flow[1] from vertex s to vertex t in $\mathcal{N}_G(q)$. Then, for each $e = \{v_i, v_j\} \in E$, either zero or one unit of flow in $F(q)$ passes through the corresponding vertex e in V_G, and thus at most one of the two edges (e, v_i) and (e, v_j) is assigned one unit of flow. This induces an orientation $\Lambda_{F(q)}$ of G based on $F(q)$ as follows: If the flow in $F(q)$ from vertex e to vertex v_i equals 1 then set $\Lambda_{F(q)}(e) := (v_i, v_j)$; else if the flow in $F(q)$ from e to v_j equals 1 then set $\Lambda_{F(q)}(e) := (v_j, v_i)$; else set $\Lambda_{F(q)}(e)$ arbitrarily.

Let $f(q)$ denote the value of a maximum directed flow from vertex s to vertex t in $\mathcal{N}_G(q)$. Then:

Lemma 6: For any $q \in \{0, 1, \ldots, \Delta\}$, $f(q) \leq q \cdot n$.

Proof: The sum of all edge capacities of edges leading into t in $\mathcal{N}_G(q)$ is $q \cdot n$. Clearly, the value of the maximum flow in $\mathcal{N}_G(q)$ cannot be larger than this sum. □

Lemma 7: For any $q \in \{0, 1, \ldots, \Delta\}$, $f(q) = q \cdot n$ if and only if $OPT(G) \geq q$.

Proof:

$\Longrightarrow$) Suppose that $f(q) = q \cdot n$ and consider the maximum flow $F(q)$ defined above. For each $v \in V$, exactly q units of flow leave the corresponding vertex v in V_G because the edge capacity of (v, t) is q and there are n such vertices. This implies that q units of flow enter v, which is only possible if there are q edges of the form (e, v) in E_G that have been assigned one unit of flow each. Therefore, the induced orientation $\Lambda_{F(q)}$ ensures that $d_{\Lambda_{F(q)}}(v) \geq q$ for every $v \in V$, which yields $OPT(G) \geq q$.

$\Longleftarrow$) Suppose that $OPT(G) \geq q$ and let Λ be a max-min orientation of G. Let F' be the following directed flow from s to t in $\mathcal{N}_G(\Delta)$:

$$F'(a) = \begin{cases} 1, & \text{if } a = (s, e) \text{ with } e \in E; \\ 1, & \text{if } a = (e, v_i) \text{ with } e = \{v_i, v_j\} \in E \\ & \quad \text{and } \Lambda(e) = (v_i, v_j); \\ 0, & \text{if } a = (e, v_i) \text{ with } e = \{v_i, v_j\} \in E \\ & \quad \text{and } \Lambda(e) = (v_j, v_i); \\ d_\Lambda(v), & \text{if } a = (v, t) \text{ with } v \in V. \end{cases}$$

For every $v \in V$, the flow in F' along the edge (v, t) in $\mathcal{N}_G(\Delta)$ is $d_\Lambda(v) \geq OPT(G) \geq q$. By reducing each such edge flow to q, one obtains a directed flow which obeys the (stricter) edge capacity constraints of the flow network $\mathcal{N}_G(q)$ and has flow value $n \cdot q$. Thus, there exists a maximum directed flow from s to t in $\mathcal{N}_G(q)$ with value $q \cdot n$, so $f(q) \geq q \cdot n$. It follows from Lemma 6 that $f(q) = q \cdot n$. □

Lemmas 6 and 7 suggest the algorithm for unweighted MAXMINO named Algorithm Exact-1-MaxMinO.

Theorem 8: Exact-1-MaxMinO solves unweighted MAXMINO in $O(m^{3/2} \cdot \log m \cdot \log^2 \Delta)$ time. □

Proof: The correctness of Exact-1-MaxMinO is guaranteed by Lemmas 6 and 7. For any $q \in \{0, 1, \ldots, \Delta\}$, to compute a maximum flow in the flow network $\mathcal{N}_G(q)$ takes $O(m^{3/2} \cdot \log m \cdot \log \Delta)$ time with the algorithm of Goldberg

1. Since all edge capacities are integers, we may assume by the integrality theorem (see, e.g., [6]) that the flow along each edge in $F(q)$ found by the algorithm in [11] is an integer.

Algorithm 1 Algorithm Exact-1-MaxMinO

1: Construct $\mathcal{N}_G$.
2: Use binary search on q in the interval $\{0, 1, \ldots, \Delta\}$ to find the integer q such that $f(q) = q \cdot n$ and $f(q+1) < (q+1) \cdot n$.
3: Compute $F(q)$ as a maximum directed flow from s to t in $\mathcal{N}_G(q)$.
4: Return $\Lambda_{F(q)}$.

and Rao [11] because $\mathcal{N}_G(q)$ contains $m + n + 2 = O(m)$ vertices and $3m + n = O(m)$ edges and the capacity of each edge in $\mathcal{N}_G(q)$ is upper-bounded by Δ. Algorithm Exact-1-MaxMinO can therefore be implemented to run in $O(m^{3/2} \cdot \log m \cdot \log^2 \Delta)$ time. □

Finally, we outline how Exact-1-MaxMinO can be applied to solve weighted MAXMINO. Let X be the set of all edges in E with weight larger than w_{min}. First modify the flow network $\mathcal{N}_G(q)$ to set $cap_q(a) = \lceil w(e)/w_{min} \rceil$. for every edge $a \in E_G$ of the form $a = (s, e)$. Then, run Exact-1-MaxMinO a total of $2^{|X|}$ times while testing all possible ways of setting the capacity of exactly one of (e, v_i) and (e, v_j) in $\mathcal{N}_G(q)$ to $w(e)$ and the other to 0 for each $e \in X$, using binary search on q in the interval $\{0, 1, \ldots, \lceil W/n \rceil\}$, and select the best resulting orientation. The asymptotic running time becomes the same as that of Exact-1-MaxMinO multiplied by $2^{|X|}$ and with an increase due to the larger interval for the binary search on q and the edge capacities being upper-bounded by $\max\{w_{max}, W/n\}$ instead of Δ.

Theorem 9: Weighted MAXMINO can be solved in $O(m^{3/2} \cdot \log m \cdot \log(w_{max} + W/n) \cdot \log(W/n) \cdot 2^{|X|})$ time, where $X = \{e \in E \mid w(e) > w_{min}\}$.

Corollary 1: If $|X| = O(\log n)$ then weighted MAXMINO can be solved in polynomial time. □

4. A simple approximation algorithm for general cases

Here, we prove that ignoring the edge weights of the input graph and applying Exact-1-MaxMinO on the resulting unweighted graph immediately yields a $\frac{w_{max}}{w_{min}}$-approximation algorithm for the general case of the problem. The algorithm is named Approximate-MaxMinO and is listed in Algorithm 2.

Algorithm 2 Algorithm Approximate-MaxMinO

1: Let G' be the undirected graph obtained from G by replacing the weight of every edge by 1.
2: Apply Algorithm Exact-1-MaxMinO on G' and let Λ' be the obtained orientation.
3: Return Λ'.

Theorem 10: Approximate-MaxMinO runs in $O(m^{3/2} \cdot \log m \cdot \log^2 \Delta)$ time and is a $\frac{w_{max}}{w_{min}}$-approximation algorithm for MAXMINO.

Proof: The asymptotic running time of Algorithm Approximate-MaxMinO is the same as that of Exact-1-MaxMinO.

To analyze the approximation ratio, observe that $\delta_\Lambda(G) \geq w_{min} \cdot \delta_\Lambda(G')$ for any orientation Λ of G because the weight of any edge in G is at least w_{min} times larger than its weight in G'. Similarly, $w_{max} \cdot \delta_\Lambda(G') \geq \delta_\Lambda(G)$ for any orientation Λ of G. Now, let Λ' be the optimal orientation for G' returned by Approximate-MaxMinO and let Λ^* be an optimal orientation for G. Note that $\delta_{\Lambda'}(G') \geq \delta_{\Lambda^*}(G')$. Thus, $\delta_{\Lambda'}(G) \geq w_{min} \cdot \delta_{\Lambda'}(G') \geq w_{min} \cdot \delta_{\Lambda^*}(G') \geq \frac{w_{min}}{w_{max}} \cdot \delta_{\Lambda^*}(G) = \frac{w_{min}}{w_{max}} \cdot OPT(G)$. □

5. An exact algorithm for cactus graphs

In this section, we present a polynomial time algorithm which obtains optimal orientations for cactus graphs. A graph is a *cactus* if every edge is part of at most one cycle. To this end, we introduce vertex weight $\alpha_G(v)$ for each vertex v in a graph G which is considered as 0 in the input graph (we omit the subscript G of $\alpha_G(v)$ if it is apparent). Here we define the notion of weighted outdegree for a vertex in a vertex and edge weighted graph. The *weighted outdegree* $d_\Lambda(v)$ of a vertex v is defined as the weight of v itself plus the total weight of outgoing arcs of v, i.e.,

$$d_\Lambda(v) = \alpha(v) + \sum_{\substack{\{u,v\} \in E: \\ \Lambda(\{u,v\})=(v,u)}} w(\{u, v\}).$$

In a cactus graph, a vertex in a cycle is a *gate* if it is adjacent to any vertex that does not belong to the cycle. Note that the unweighted degree of a gate is at least three. As for the number of gates in a cycle, the following is known:

Proposition 11 (Proposition 2 in [2]): In a cactus graph G in which $deg(v) \geq 2$ for every vertex v, there always exists a cycle with at most one gate.

The main part of the proposed algorithm Exact-Cactus-MaxMinO is shown in Algorithm 3, which solves the decision version of the problem MAXMINO: Given a number K, this problem asks whether there exists an orientation whose value is at least K. We can develop an algorithm for the original problem MAXMINO by using this algorithm $O(\log \Delta)$ times in a binary search manner on optimal value, which is upper-bounded by Δ.

The correctness of Exact-Cactus-MaxMinO is based on the following property on optimal orientations for two graphs.

Proposition 12: Consider two graphs G and G' that differ only on their vertex weights. If $\alpha_G(v) \leq \alpha_{G'}(v)$ for every vertex v, then $OPT(G) \leq OPT(G')$ holds. □

Algorithm 3 Algorithm Exact-Cactus-MaxMinO

1: **repeat**
2: For a vertex v,
3: **if** $\alpha(v) + d(v) < K$ **then**
4: output No and halt.
5: **else if** $deg(v) = 1$ **then**
6: (let its connecting edge be $e = \{v, u\}$)
7: **if** $\alpha(v) < K$ **then**
8: $\Lambda(e) := (v, u)$
9: **else**
10: $\Lambda(e) := (u, v)$ and increase $\alpha(u)$ by $w(e)$
11: **end if**
12: Remove v and e.
13: **else if** $deg(v) = 2$ **then**
14: (let $e_1 = \{p, v\}$ and $e_2 = \{v, q\}$)
15: **if** $\alpha(v) + w(e_1) < K$ and $\alpha(v) + w(e_2) < K$ **then**
16: $\Lambda(e_1) := (v, p)$ and $\Lambda(e_2) := (v, q)$. Remove v, e_1, and e_2.
17: **else if** $\alpha(v) + w(e_1) < K$ and $\alpha(v) + w(e_2) \geq K$ **then**
18: $\Lambda(e_1) := (p, v)$ and $\Lambda(e_2) := (v, q)$ and also increase $\alpha(p)$ by $w(e_1)$. Remove v, e_1, and e_2.
19: **end if**
20: **end if**
21: **until** there does not exist a vertex v satisfying either one of the above conditions
22: **for all** $C := \langle v_0, v_1, \cdots, v_\ell = v_0 \rangle$ that has at most one gate **do**
23: **if** C does not have a gate **then**
24: $\Lambda(\{v_i, v_{i+1}\}) := (v_i, v_{i+1})$ for $0 \leq i \leq \ell - 1$.
25: Remove C.
26: **else**
27: Let v_0 be the gate.
28: **if** there exists a vertex v_j, $j \neq 0$ satisfying $\alpha(v_j) \geq K$ in C **then**
29: Assign $\Lambda(\{v_i, v_{i+1}\}) := (v_i, v_{i+1})$ for $0 \leq i \leq j - 1$ and $\Lambda(\{v_i, v_{i+1}\}) := (v_{i+1}, v_i)$ for $j \leq i \leq \ell - 1$. Increase $\alpha(v_0)$ by $w(\{v_0, v_1\}) + w(\{v_0, v_{\ell-1}\})$.
30: **else**
31: If $w(\{v_0, v_1\}) > w(\{v_0, v_{\ell-1}\})$ then assign $\Lambda(\{v_i, v_{i+1}\}) := (v_i, v_{i+1})$ for $0 \leq i \leq \ell - 1$ and increase $\alpha(v_0)$ by $w(\{v_0, v_1\})$, otherwise $\Lambda(\{v_i, v_{i+1}\}) := (v_{i+1}, v_i)$ for $0 \leq i \leq \ell - 1$ and increase $\alpha(v_0)$ by $w(\{v_0, v_{\ell-1}\})$.
32: **end if**
33: Remove C except the gate v_0.
34: **end if**
35: **end for**
36: **if** the graph is empty **then**
37: output Λ and halt.
38: **else**
39: go back to line 1.
40: **end if**

Theorem 13: Given a cactus graph G and a target K, Exact-Cactus-MaxMinO outputs an orientation Λ such that $\delta_\Lambda(G) \geq K$ if such an orientation exists, in polynomial time.

Proof: First we estimate the running time. Since each of executions of **repeat** loop or **for all** loop determines the direction of at least one edge, the total number of times those steps are being processed is bounded by $O(m)$. Also all of those steps can be done in $O(m)$ time because they only find a vertex or a cycle with testing certain conditions by a constant time. Hence the total running time is $O(m^2)$. It can be reduced to $O(m + n \log n)$ by a careful implementation of the algorithm, but we omit the details here.

Next we show the correctness of the algorithm. Each execution that removes some vertices and edges from the current graph H (lines 12, 16, 18, 25 and 33) may increase the weight of a remaining vertex, and then obtains a modified graph H'. What we would like to show is that if $OPT(H) \geq K$, then (i) also $OPT(H') \geq K$, (ii) the determined directions of edges are correct, and so (iii) all the vertices removed at the step have weighted outdegree at least K. Assume that $OPT(H) \geq K$.

Lines 3-4: If the condition is satisfied, the answer is clearly No.

Lines 5-12: It holds that $\alpha(v) + w(e) \geq K$ since it passes through the check in line 3. In the case that $\alpha(v) < K$, if we assign $\Lambda(e) := (u, v)$, then the weighted outdegree of v is less than K, which contradicts the assumption $OPT(H) \geq K$. Hence the assignment $\Lambda(e) := (v, u)$ is correct and also the removed vertex v has weighted outdegree at least K. Also it holds that $OPT(H') \geq OPT(H)$, otherwise, it contradicts that $OPT(H)$ is the optimal value.

Let us consider the other case that $\alpha(v) \geq K$. The weighted outdegree of the removed vertex v is at least $\alpha(v) \geq K$ in this case whichever the direction assigned to the edge e is. There are two possibilities: We assign either $\Lambda(e) := (u, v)$ or $\Lambda(e) := (v, u)$. Let the graph obtained by the former assignment with increasing $\alpha(u)$ by $w(e)$ be H', and let the graph obtained by the latter be H''. From Proposition 12, we observe that $OPT(H') \geq OPT(H'')$. If $OPT(H'') \geq OPT(H)$, then it holds that $OPT(H') \geq OPT(H'') \geq OPT(H)$ for both of the directions of the edge e. Conversely, the inequality $OPT(H'') < OPT(H)$ means that the weighted outdegree of the vertex u must be augmented by orienting the edge $e = \{v, u\}$ as (u, v) in the optimal orientation for H, because, otherwise it contradicts that $OPT(H)$ is the optimal value. Thus the assignment $\Lambda(e) := (u, v)$ is correct and it holds that $OPT(H') \geq OPT(H) \geq K$.

Lines 13-20: If we do not follow the rules here, the weighted outdegree of the processed vertex would be less than $K \leq OPT(H)$, which implies the operations in these lines are correct. As a result, the weighted outdegree of the removed vertex v at line 16 (resp., line 18) is at least K from the assumption that v does not satisfy the condition of line

3 (resp., line 15), and also $OPT(H') \geq OPT(H) \geq K$.

Lines 22-35: First of all, at the beginning of this part, it holds that $deg(v) \geq 2$ for every vertex v since it passes lines 5-12. From Proposition 11, we can always find a cycle having at most one gate.

Lines 23-25: It is obvious that the vertices removed at this step have weighted outdegree at least K because they passed lines 13-19. In addition to that it holds that $OPT(H') \geq OPT(H)$ since C is a connected component in H and $OPT(H) = \min\{OPT(C), OPT(H')\}$.

Lines 27-33: The vertices removed in this step all have weighted outdegree at least K because of the conditions of lines 13-19. Also the proposed assignment of directions for the cycle C increases $\alpha(v_0)$ as much as possible without breaking optimality, and it derives $OPT(H') \geq K$ by a similar argument as the one used in lines 5-12.

By the above discussion, if all the vertices are removed without answering No, the weighted outdegree of every vertex by the orientation Λ is at least K. □

From Theorem 13, we can solve MAXMINO for cactus graphs in polynomial time by using EXACT-CACTUS-MAXMINO as an engine of the binary search.

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